Exact relativistic treatment of stationary black-hole–disk systems

C. Klein*
LUTh, Observatoire de Paris, 92195 Meudon Cedex, France
(Received 19 January 2003; published 24 July 2003)

We present and discuss a class of exact solutions to the stationary axisymmetric Einstein equations which describe a regular black hole surrounded by a disk of infinite extension with finite inner radius. The spacetimes are asymptotically flat and equatorially symmetric. They form a subclass of Korotkin's solutions to the Ernst equation on partially degenerate hyperelliptic Riemann surfaces of genus $g+2$. The metric is given explicitly in terms of theta functions on a surface of genus g. The case of genus zero when the solutions are given in terms of elementary functions is discussed in detail.

DOI: 10.1103/PhysRevD.68.027501 PACS number(s): 04.20.Jb, 02.30.Jr

Systems of black holes and surrounding thin disks of collisionless matter, so called dust, are discussed in astrophysics as models for accretion disks around black holes and galaxies with supermassive black holes in the nucleus, see e.g. [1–3] for the observational evidence and [4–6] for the theoretical background. Since a black hole is a genuinely relativistic object, a fully relativistic treatment of such situations is necessary. In cases where the mass of the disk is not negligible compared to the mass of the black hole, the self-gravity of the disk plays a role. Exact solutions to the Einstein equations for black holes with disks could provide insight in the mathematical structure of the equations and the underlying physics. In addition they could be used to test numerical codes for these models. Since disks of pressureless matter are known to be in general unstable, the solutions could be used as exact initial data for numerical simulations of gravitational collapse.

Disks around black holes must have an inner radius strictly larger than the photon radius, the radius where photons move on a sphere around the black hole, to exclude superluminal velocities of the particles. Since the stationary axisymmetric Einstein equations in the form of Ernst [7] are completely integrable [8,9], powerful solution techniques are at hand to obtain explicit solutions for non-active systems consisting of a black hole and an infinitesimally thin disk. In this context Riemann surface techniques in the form of Korotkin's theta functional solutions to the Ernst equation [10,11] seem to be the most promising. They were already successfully applied to give explicit solutions for the pure disk case [12,13].

The practical importance of methods from the theory of integrable systems in the generation of solutions to the Einstein equations is somewhat limited by the fact that they work *a priori* only locally. This means they establish solutions in finite regions of spacetime, but do not guarantee global regularity in the exterior of prescribed sources. In the context of algebro-geometric solutions, the integrable nonlinear equation is linearized on the Jacobian of a plane algebraic curve which implies that different solutions can be combined in a non-linear way (sometimes called "non-linear superposition"). But since this is a non-linear operation, the combined solutions will have in general singularities as singular rings, Weyl struts on the axis and singular horizons. A well known example is the "superposition" of two Kerr black holes which results in a spacetime with a Weyl strut on the axis between the holes, see e.g. [19]. Such techniques are thus only helpful if the analytical properties of the solutions can be obtained in a general way as in [15].

In this paper we discuss a subclass of Korotkin's solutions which are obtained on partially degenerate Riemann surfaces where precise statements on the analyticity of the solutions can be made. The solutions describe annular disks of infinite extension but finite mass and inner radius $\rho_0=1$ in the equatorial plane. The black hole in the center of the annulus is characterized by a regular Killing horizon. The solutions are asymptotically flat, equatorially symmetric and regular outside the horizon and the disk. They contain a free function and a set of free parameters, the branch points of the Riemann surface. If the energy conditions are satisfied, the matter in the disk can be interpreted as in [14] as made up of one or more components of collisionless matter. Since the spacetimes are asymptotically flat, the matter in the asymptotic region behaves as in [14] as free particles which move on Keplerian orbits. The approach presented here makes it possible to map the problem for an infinite disk around a black hole to the problem of a disk of finite radius which allows the use of the techniques in [15,16] in this context. For the class of solutions presented here which can be seen as a combination of the Kerr solution with an infinite disk, we are thus able to establish regularity of the horizon and the exterior of the disk.

NEWTONIAN CASE

It is instructive to consider first a Newtonian analogue for a system consisting of a black hole and a disk, i.e. a point mass with a surrounding annular disk. In Newtonian theory gravity is described by a scalar potential U which is in vacuum a solution to the Laplace equation $\Delta U=0$. We use cylindrical coordinates ρ,ζ,ϕ and consider only the axisymmetric case. It is known (see e.g. [16]) that disks in the equatorial plane $\zeta=0$ between radii 0 and 1 can be described by a potential of the form

$$U(\rho,\zeta)=-\frac{1}{4\pi i}\int_{\Gamma}\frac{\mathcal{A}(\tau)d\tau}{\sqrt{(\tau-\zeta)^2+\rho^2}}, \qquad (1)$$

*Current address: Max-Planck-Institut für Physik, Föhringer Ring 6, 80805 München, Germany.

where Γ is the part of the imaginary axis between $-i$ and i on the first sheet of the Riemann surface $\mathcal{L}_0$ defined by the algebraic relation $\mu_0^2(K)=(K-\zeta)^2+\rho^2$; $\mathcal{A}$ is a Hölder continuous function on Γ independent of the physical coordinates. It has to vanish at the end points of Γ to avoid a ring singularity at the rim of the disk. Relation (1) can be used to construct an annular disk of infinite extension with a finite inner radius with the help of a Kelvin transformation as in [17,18], a reflection at the inner circle in the ρ,ζ plane. It is a property of the Laplace operator that a function $\hat{U}=U(\rho/r,\zeta/r)/r$ with $r^2=\rho^2+\zeta^2$ is a solution to the Laplace equation if $\Delta U(\rho,\zeta)=0$. Using the linearity of the Laplace equation and introducing the complex coordinate $\xi=\zeta-i\rho$ we get that

$$U(\xi,\bar{\xi})=-\frac{m}{r}-\frac{1}{4\pi ir}\int_\Gamma \frac{\mathcal{A}(\tau)}{\sqrt{(\tau-1/\xi)(\tau-1/\bar{\xi})}}d\tau \quad (2)$$

is a solution to the Laplace equation which describes a point mass surrounded by a disk of infinite outer radius and of inner radius 1. The function $\mathcal{A}$ must satisfy the additional condition $\mathcal{A}(0)\neq 0$. It can be easily seen that the solution U is equatorially symmetric if $\mathcal{A}$ is an even function. We note for later use that the integral in expression (2) can be written in the form

$$U(\rho,\zeta)=-\frac{1}{4\pi i}\int_{\Gamma_\infty}\frac{\ln G(\tau)d\tau}{\sqrt{(\tau-\zeta)^2+\rho^2}} \quad (3)$$

after a transformation $\tau\to 1/\tau$, where $\Gamma_\infty=(-i\infty,-i]\cup[i,i\infty)$ and $\mathcal{A}(\tau)=\ln G(1/\tau)/\tau$ (the integral has to be understood as a standard contour integral in the complex plane after the appropriate choice of the sign). This establishes the relation to the form of the solutions discussed in [15].

The total mass M of the system (2) is given by $M=m+\mathcal{A}(0)/4$. Thus the contribution of the disk to the mass is due to the value of $\mathcal{A}$ at the origin. We assume that the matter in the disk consists of pressureless matter rotating with angular velocity Ω. Thus the centrifugal force is the only force to stabilize the disk against gravitational collapse and has to compensate the gravitational attraction, $U_\rho=\Omega^2(\rho)\rho$. In this Newtonian setting the velocity $\Omega\rho$ of particles with radius ρ in the disk is obviously related to the point mass. By limiting the value of m one can thus impose upper limits on the velocities of the particles. Asymptotically the angular velocity reads $\Omega^2=M/\rho^3$, the Keplerian value for test particles.

RELATIVISTIC CASE

In the stationary axisymmetric vacuum, the metric can be written in the Weyl-Lewis-Papapetrou form (see [19])

$$ds^2=-e^{2U}(dt+a\,d\phi)^2+e^{-2U}[e^{2k}(d\rho^2+d\zeta^2)+\rho^2d\phi^2], \quad (4)$$

where ∂_t and ∂_ϕ are the two commuting asymptotically timelike, respectively, spacelike Killing vectors. The Einstein equations are in this case equivalent to the Ernst equation [7]

$$\mathcal{E}_{\xi\bar{\xi}}-\frac{1}{2(\bar{\xi}-\xi)}(\mathcal{E}_{\bar{\xi}}-\mathcal{E}_\xi)=\frac{2\mathcal{E}_\xi\mathcal{E}_{\bar{\xi}}}{\mathcal{E}+\bar{\mathcal{E}}}, \quad (5)$$

where $\mathcal{E}=e^{2U}+ib$; the real function b is related to the metric functions via a duality rotation, $b_\xi=-(i/\rho)a_\xi e^{4U}$. It is a consequence of the Einstein equations that the complete metric can be obtained for a given Ernst potential in terms of quadratures.

Since the Ernst equation is completely integrable, methods from soliton theory can be applied to generate solutions. The equation being elliptic it obviously does not have wave-like or solitonic solutions, but in a mathematical sense the Kerr solution for the exterior of a rotating black hole corresponds to a 2-soliton. The Ernst potential can be written in this case in the form

$$\mathcal{E}=\frac{e^{-i\varphi}r_1+e^{i\varphi}r_2-2m\cos\varphi}{e^{-i\varphi}r_1+e^{i\varphi}r_2+2m\cos\varphi} \quad (6)$$

with $r_i=\mu_0(K_i)(i=1,2)$ and $K_1=-K_2=-m\cos\varphi$. The solution is parametrized by the two parameters m (positive definite) and φ $(0\leq\varphi\leq\pi/2)$ which can conveniently be related to the asymptotically defined Arnowitt-Deser-Misner (ADM) mass $M=m$ and the angular momentum $J=m^2\sin\varphi$. The horizon of the black hole is located in Weyl coordinates on the axis $\rho=0$ between $-m\cos\varphi$ and $m\cos\varphi$. The real part of the Ernst potential vanishes at the end points of the horizon where the ergosphere touches the axis and is negative in between. For $\varphi=0$ the solution becomes static, the spherically symmetric Schwarzschild solution. For $\varphi=\pi/2$, the case of the extreme Kerr solution, the horizon degenerates.

Due to the complete integrability of the Ernst equation, rich classes of solutions can be constructed in terms of theta functions on Riemann surfaces. Korotkin [10] considered families of hyperelliptic surfaces $\mathcal{L}$ of genus $g+2$ defined by the algebraic relation

$$\mu^2(K)=(K-\xi)(K-\bar{\xi})\prod_{i=1}^{g+2}(K-E_i)(K-\bar{E}_i). \quad (7)$$

The main difference to corresponding solutions to other integrable equations like the Korteweg–de Vries equation (see e.g. [20]) is the dependence of some of the branch points on the physical coordinates (the E_i are constant with respect to ρ,ζ). This implies that the solutions in terms of theta functions here will be neither periodic nor quasi-periodic but can be asymptotically flat. The Ernst potential can be written in the form (see also [15])

$$\mathcal{E}=\frac{\Theta_{pq}[\omega(\infty^+)+u]}{\Theta_{pq}[\omega(\infty^-)+u]}e^I, \quad (8)$$

where

$$\Theta_{pq}(z)=\sum_{m\in\mathbb{Z}^{g+2}}\exp\left\{\frac{1}{2}\langle\Pi(p+m),(p+m)\rangle+\langle p+m,z+2\pi iq\rangle\right\} \quad (9)$$

is the theta function with characteristic $p,q\in\mathbb{C}^{g+2}$ on the surface $\mathcal{L}$; $\omega(Q)$ is the Abel map with base point ξ, Π is the matrix of b periods, a point on $\mathcal{L}$ is denoted by $Q^{\pm}$ where $\pm$ indicates the sheet (for details see e.g. [15]). Since we want to consider disks of infinite extension with inner radius 1, we consider the straightforward generalization of the Newtonian formulas, i.e.

$$I=\frac{1}{2\pi i}\int_{\Gamma_\infty}\ln G d\omega_{\infty^+\infty^-}, \tag{10}$$

where $d\omega_{AB}$ is the normalized differential of the third kind with simple poles in A and B with residues $+1$, respectively, -1; the vector u of its b periods can be written in the form $u=(1/2\pi i)\int_{\Gamma_\infty}\ln G d\omega$.

In [21] it was shown that the Kerr solution can be obtained from Eq. (8) for even g in the absence of a disk ($u_i=I=0$) in the limit $E_{g+j}\to\bar{E}_{g+j}=K_j$, $j=1,2$ with $K_1=-K_2=m\cos\varphi$. The additional branch points are subject to the condition $E_i=-\bar{E}_{g+1-i}$, $i=1,\dots,g/2$, and the characteristic reads

$$\begin{bmatrix} 0 & \dots & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \dots & \frac{1}{2} & -\frac{1}{4}+\frac{\ln A}{2\pi i} & \frac{1}{4}-\frac{\ln A}{2\pi i}\end{bmatrix}, \tag{11}$$

where with A real

$$e^{-i\varphi}=\frac{iA+1}{iA-1}\prod_{i=1}^{g}\sqrt{\frac{K_1-F_i}{K_1-E_i}}. \tag{12}$$

This generalizes a result in [10] where the Kerr solution was obtained in the above setting for $g=0$ in the "solitonic limit." Here the Kerr solution is obtained by partially degenerating the surface $\mathcal{L}$ which leads to an Ernst potential given on a surface $\tilde{\mathcal{L}}$ of genus g where the cuts $[E_{g+j},\bar{E}_{g+j}]$, $j=1,2$ are removed from $\mathcal{L}$. The theta function on $\mathcal{L}$ is in this limit proportional to

$$\begin{aligned}\mathcal{F}(x)=&\theta_{pq}\left(\tilde{x}+\int_{K_2^-}^{K_1^+}d\tilde{\omega}\right)+e^{x_{g+1}+x_{g+2}}\theta_{pq}\left(\tilde{x}+\int_{K_1^+}^{K_2^-}d\tilde{\omega}\right)\\ &-ie^{-P}\left[Ae^{x_{g+1}}\theta_{pq}\left(\tilde{x}+\int_{K_1^+}^{K_2^+}d\tilde{\omega}\right)\right.\\ &\left.-A^{-1}e^{x_{g+2}}\theta_{pq}\left(\tilde{x}+\int_{K_2^+}^{K_1^+}d\tilde{\omega}\right)\right],\end{aligned} \tag{13}$$

where $P=\Pi_{(g+1)(g+2)}$, i.e. it can be expressed completely in terms of quantities defined on $\tilde{\mathcal{L}}$, the corresponding theta function θ, and the differentials $d\tilde{\omega}$. This makes it possible to combine the Kerr solution and the disks of infinite extension in a non-linear way by considering the above limit of the Ernst potential (8) which leads in the used notation to the main result of this paper. The important point is that we can establish the regularity of the horizon and the range of the free parameters where the exterior of the disk is regular in a general way. Exactly these features which are summarized in the following theorem establish the physical relevance of this class of solutions.

Theorem. The potential

$$\begin{aligned}\mathcal{E}&=\frac{\mathcal{F}(u+\omega(\infty^+))}{\mathcal{F}(u+\omega(\infty^-))}e^{\tilde{I}+\omega_{g+1}(\infty^+)+\omega_{g+2}(\infty^+)},\\ \tilde{I}&=\frac{1}{2\pi i}\int_{\Gamma_\infty}\ln G d\tilde{\omega}_{\infty^+\infty^-},\end{aligned} \tag{14}$$

where $d\omega_{g+j}=d\omega_{K_j^-K_j^+}$, $j=1,2$, is an equatorially symmetric solution to the Ernst equation which has a finite jump at the disk between 1 and infinity in the equatorial plane and a regular Killing horizon on the axis between $-m\cos\varphi$ and $m\cos\varphi$. The solution is regular in the exterior of the horizon and the disk if $\mathcal{F}(u+\omega(\infty^-))\neq 0$.

The condition $\mathcal{F}(u+\omega(\infty^-))\neq 0$ just reflects the fact that within general relativity, arbitrary amounts of energy cannot be concentrated in a finite region of spacetime without leading to a black hole or a singularity. In the present setting, the formation of singularities is just controlled by this condition. It determines for a given solution the range of the physical parameters where the solution is regular in the exterior of the horizon and the disk, for the solution [12] see [22].

We briefly sketch the proof. The fact that Eq. (14) solves the Ernst equation is a direct consequence of working out the limit of Eq. (8) as in [21]. The behavior at the disk was demonstrated in [15]; here the discontinuity is located in the equatorial plane between 1 and infinity because of the Kelvin transformation. The potential on the axis can be calculated as in [15] by a further degeneration of the Riemann surface $\tilde{\mathcal{L}}$: since the axis is the limit $\xi\to\bar{\xi}$, the potential can be expressed in terms of theta functions on the surface $\mathcal{L}'$ where the cut $[\xi,\bar{\xi}]$ is removed from $\tilde{\mathcal{L}}$. These formulas also hold on the horizon where it is a consequence of [21] that the metric is identical to the Kerr metric which establishes the regularity of the horizon. All metric functions being given explicitly (see [11,15]), the same is true on the axis where one finds that the function e^{2U} is zero at the end points of the horizon where the ergosphere touches the horizon and is negative in between. The functions a and k are zero on the axis in the exterior of the horizon and have non-vanishing constant values with respect to ζ on the horizon which characterizes a regular horizon (see e.g. Carter in [5]). In the rest of the spacetime the Ernst potential is regular unless there is a zero in the denominator of Eq. (14) as was shown in [15].

Since the disk extends to infinity, the spacetime corresponding to Eq. (14) is in general not asymptotically flat. Choosing $\mathcal{A}(0)$ to be finite, one assures that $\mathcal{E}$ is finite at infinity. If $\mathcal{F}(\tilde{u})$ does not vanish on the regular part of the axis where it is a constant, the Ernst potential tends to 1 and one can define in a standard way multipole moments on the axis, $\mathcal{E}=1-2M/\zeta-2J/\zeta^2+\dots$. For a general choice of the function G and the additional parameters, the mass M will not be real which implies a Newman-Unti-Tamburino (NUT) parameter which is believed to be unphysical. The condition to have a real mass yields for $g>0$ a quadratic relation for the parameter A if the function $\mathcal{A}$ and the remaining parameters are given. For parameters chosen in a way that this relation can be satisfied by a real A, the spacetime is asymptotically flat in the standard sense. This procedure implies that the rotation state of the hole and the disk must be synchronized. In other words the black hole and the disk cannot rotate in an arbitrary way in an asymptotically flat setting.

The case $g=0$: For illustration we will consider the simplest case $g=0$ in more detail even though it will turn out that the disks always consist of exotic matter in this case. Since the Ernst potential is defined on a surface of genus zero it can be expressed in terms of elementary functions. Writing the Ernst potential in the form $\mathcal{E}=(\mathcal{G}-1)/(\mathcal{G}+1)e^{I}$, formula (14) takes the form

$$\mathcal{G}=\frac{1+e^{u_1+u_2}}{1-e^{u_1+u_2}-i(Ae^{u_1}+e^{u_2}/A)}\frac{r_1-r_2}{E_1-E_2}+\frac{i(Ae^{u_1}-e^{u_2}/A)}{1-e^{u_1+u_2}-i(Ae^{u_1}+e^{u_2}/A)}\frac{r_1+r_2}{E_1-E_2}. \tag{15}$$

We put $A=\cot(\varphi/2)$ to obtain for $u=I=0$ the Kerr solution in the form (6). The static limit $\varphi=0$ of these solutions describes the superposition of a Schwarzschild black hole and an annular disk as in [17,18]. Since this is a solution to the Laplace equation in the exterior of the horizon and the disk, the solution is always regular there and asymptotically flat. This implies that the solutions (15) are also regular in this sense for small φ; it depends, however, on the choice of the function $\mathcal{A}$ whether the extreme limit $\varphi\to\pi/2$ can be reached without generating naked singularities as in the case of the over-extreme Kerr solutions. In non-static situations the spacetime will have a NUT parameter unless

$$\int_{-i}^{i}\frac{\mathcal{A}(\tau)d\tau}{\tau^2m^2\cos^2\varphi-1}=0. \tag{16}$$

This is obviously only possible if $\mathcal{A}$ changes sign on the path of integration. The ADM mass is given by $M=m+\mathcal{A}(0)/4$, the angular momentum by $J=m^2\sin\varphi[1+\mathcal{A}(0)/(2m)]$. The values for the invariant surface $\mathcal{A}_{BH}$ of the horizon and the constant value $1/\Omega_{BH}$ of the metric function a at the horizon are unchanged with respect to the pure Kerr case. The constant Ω_{BH} can be interpreted as the angular velocity of the black hole (see e.g. [5]).

Since the density in the Newtonian limit is proportional to $\int_\rho^1\mathcal{A}(it)/\sqrt{t^2-1}$ (see [16]), it must in this case also change sign. Numerically one finds that in the relativistic case, too, there must be regions with negative density in the disk which implies that the case $g=0$ does not lead to asymptotically flat spacetimes with non-exotic matter in the disk.

OUTLOOK

In contrast to the genus 0 case, the condition of a vanishing NUT parameter can be satisfied for higher genus by choosing the parameter A appropriately in dependence of the other parameters whereas the density can be positive in the whole disk. For a given function $\mathcal{A}$, the parameter m has to be chosen in a way that the energy conditions in the disk are satisfied. The analytically known expressions in terms of theta constants for the horizon surface $\mathcal{A}_{BH}$ and the angular velocity Ω_{BH} are in general different from the pure Kerr case.

Thus physically acceptable disks can in principle be found for $g\geqslant 2$. The absence of non-static solutions for black holes with surrounding disks of non-exotic matter in this formalism for genus 0 is not surprising since the simplest stationary pure disk solutions [12,13] were also given on genus 2 surfaces. Since it is straightforward to extend the formalism of [16] and the numerical treatment of theta functions in [22] to the case discussed here, it should be possible to identify physically interesting black hole disk systems in this class. Whether the algebraic problems in solving boundary value problems due to a prescribed matter distribution in the disk can be handled analytically will be the subject of further research.

I thank B. Carter, J. Frauendiener, E. Gourgoulhon, J.-M. Huré, and D. Korotkin for helpful discussions and hints. This work was supported by the Marie Curie program of the European Union.

[1] M. Rees, in *Black Holes and Relativistic Stars*, edited by R. Wald (University of Chicago Press, Chicago, 1998), p 79.

[2] A. Laor, Phys. Rep. **311**, 451 (1999).

[3] *Observational Evidence for the Black Holes in the Universe*, edited by S. Chakrabarti (Kluwer, Dordrecht, 1999).

[4] J. Binney and S. Tremaine, *Galactic Dynamics* (Princeton University Press, Princeton, 1987).

[5] *Black Holes*, edited by C. Dewitt and B. DeWitt (Gordon and Breach, New York, 1973).

[6] O. Semerák, *Gravitation: Following the Prague Inspiration (To Celebrate the 60th Birthday of Jiri Bicak)*, edited by O. Semerak, J. Podolsky, and M. Zofka (World Scientific, Singapore, 2002), p. 111, gr-qc/0204025.

[7] F.J. Ernst, Phys. Rev. **167**, 1175 (1968); **168**, 1415 (1968).

[8] D. Maison, Phys. Rev. Lett. **41**, 521 (1978).

[9] V.A. Belinski and V.E. Zakharov, Sov. Phys. JETP **48**, 985 (1978).

[10] D.A. Korotkin, Theor. Math. Phys. **77**, 1018 (1989).

[11] D. Korotkin and V. Matveev, Funct. Anal. Appl. **34**, 1 (2000).

[12] G. Neugebauer and R. Meinel, Phys. Rev. Lett. **75**, 3046 (1995).

[13] C. Klein and O. Richter, Phys. Rev. Lett. **83**, 2884 (1999).

[14] J. Bičák and T. Ledvinka, Phys. Rev. Lett. **71**, 1669 (1993).

[15] C. Klein and O. Richter, Phys. Rev. D **58**, 124018 (1998).

[16] C. Klein, Phys. Rev. D **63**, 064033 (2001).

[17] J. Lemos and P. Letelier, Phys. Rev. D **49**, 5135 (1994).

[18] C. Klein, Class. Quantum Grav. **14**, 2267 (1997).

[19] D. Kramer, H. Stephani, E. Herlt, and M. MacCallum, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 1980).

[20] E.D. Belokolos, A.I. Bobenko, V.Z. Enolskii, A.R. Its, and V.B. Matveev, *Algebro-Geometric Approach to Nonlinear Integrable Equations* (Springer, Berlin, 1994).

[21] C. Klein, Theor. Math. Phys. (to be published).

[22] J. Frauendiener and C. Klein, Phys. Rev. D **63**, 084025 (2001).